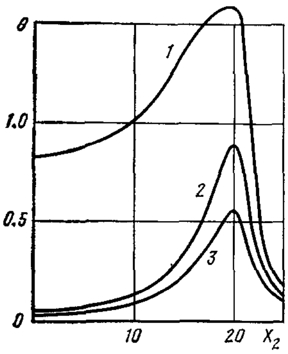
is written, apart from a constant factor, in the form

$$\int_{-a_1}^{a_1} \cos \lambda_n^{(1)} x \, \frac{6(x-x_1)^2(x_2-a_2)^2-(x_2-a_2)^4-(x-x_1)^4}{((x-x_1)^2+(x_2-a_2)^2)^3} \, dx \tag{4.1}$$

Integrating (4.1) twice by parts, a power-law singularity can be extracted that occurs at the angular point (a_1, a_2) in the form

$$\frac{(x_1-a_1)\,[(x_1-a_1)^2-(x_2-a_2)^2]}{[(x_1-a_1)^2+(x_2-a_2)^2]^2}$$

Analogous power-law singularities are obtained at angular points when investigating the other components in the expression for Ω_{ij}.



5. Results of numerical investigations. Formally setting $\alpha_1 t_c = \alpha_2 t_c = q_0$, $\alpha_1 = \alpha_2 = 0$ in (1.6) and (3.5), we will obtain the solution of the stationary heat conduction problem and the corresponding static thermo-elasticity problem for a plate with a rectangular cutout on whose boundaries the heat flux q_0 is given. For this case the dimensionless temperature field $\theta = T\lambda/q_0\delta$ was computed as a function of $X_2 = x_2/\delta$ for $A_1 = a_1/\delta = 10$, $A_2 = a_2/\delta = 20$, $\mathrm{Bi} = \alpha_3\delta/\lambda = 0.1$ and different $X_1 = x_1/\delta$. A 20×20 and 40×40 matrix of the truncated system was formed in solving system (1.7) by the method of reduction. Results of the calculations are practically identical. The results of the temperature field computations are represented as graphs in the figure for $X_1 = 10$; 11.25, 12.5 (curves *1, 2, 3,* respectively). It follows from the graphs that the maximum value of the temperature is achieved at the angular point. The temperature is equalized with distance from the boundary.

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ACTION OF A UNIFORMLY VARIABLE MOVING FORCE ON A TIMOSHENKO BEAM ON AN ELASTIC FOUNDATION. TRANSITIONS THROUGH THE CRITICAL VELOCITIES*

YU.D. KAPLUNOV and G.B. MURAVSKII

The vibrations of an infinite Timoshenko-type beam on an elastic foundation subjected to a force whose point of application moves over the beam with constant acceleration are considered. Resonance effects associated with the transition of the velocity of motion of the load through three critical values characteristic for the system being considered are studied. Asymptotic representations are constructed for the solution of the problem corresponding to the load acceleration approaching zero.

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Cases of high-velocity loads acting on a structure of large extent are encountered more and more often in engineering. In this connection it becomes of greater importance to take account of the resonance phenomena determined by the presence of so-called critical velocities in a mechanical system. If the slowly changing velocity of motion passes through a critical value, we would expect, by analogy with the well-known phenomenon of the "passage through resonance", that the growth of displacements of the points of a mechanical system would be all the more explicit the smaller the magnitude of the acceleration of the moving load. Such questions were studied in /1-4/ as they apply to certain one-dimensional systems. A Timoshenko-type beam on an elastic foundation is an example of a more complex system; it has three critical velocities, the first of which is due to beam and foundation interaction, while the other two correspond to velocities of shear and tension-compression wave propagation in the beam material.

1. Construction of the solution of the problem. Let a force P, whose point of application moves along the X axis according to the law $s(t) = v_0 t \pm wt^2/2$, be applied to an infinite Timoshenko beam resting on an elastic foundation at the time $t = 0$. Here s is the distance between the point of application of the force and the origin, $v_0 \geqslant 0$ is the initial velocity $w > 0$ is the acceleration, and the upper sign corresponds to accelerated motion, and the lower to retarded motion. We introduce the dimensionless quantities

$$\tau = t\left(\frac{k}{m}\right)^{1/2}, \quad \xi = x\left(\frac{k}{4EJ}\right)^{1/4}, \quad s_1(\tau) = v_{01}\tau \pm \frac{w_1\tau^2}{2}$$
$$v_{01} = v_0\left(\frac{m^2}{4kEJ}\right)^{1/4}, \quad w_1 = \frac{wm}{(4k^3EJ)^{1/4}}$$

where k is the coefficient of elasticity of the foundation, m is the mass per unit length of the beam, and EJ is the bending stiffness. We will represent the beam deflections in the form

$$y(\zeta, \tau) = P\left(\frac{m}{k}\right)^{1/2} \int_0^\tau u(\zeta + s_1(\tau) - s_1(\tau_1), \tau - \tau_1)\, d\tau_1 \tag{1.1}$$

where $\zeta = \xi - s_1(\tau)$ is the dimensionless coordinate of a point of the beam in the moving coordinate system, and $u(\xi, \tau)$ is the solution corresponding to the action of an instantaneous unit impulse. According to /5/

$$u(\xi, \tau) = \frac{1}{\pi(4km^2EJ)^{1/4}} \int_0^\infty \frac{\cos\lambda\xi}{2SD_1^{1/2}} (f_\alpha \sin\tau\alpha - f_\beta \sin\tau\beta)\, d\lambda \tag{1.2}$$

$$D_1 = r^2 - \frac{1}{S}(\lambda^4 R + \lambda^2 + 4R) = 4\frac{\lambda^2 R^2}{S} + \left[2\frac{R}{S} - \frac{1}{2} + \frac{\lambda^2}{2}\left(\frac{1}{S} - R\right)\right]^2$$

$$\alpha = (r - D_1^{1/2})^{1/2}, \quad \beta = (r + D_1^{1/2})^{1/2},$$
$$r = \frac{1}{2}\left[4\frac{R}{S} + 1 + \lambda^2\left(R + \frac{1}{S}\right)\right]$$
$$f_\gamma = \frac{4R - S\gamma^2 + \lambda^2}{\gamma} (\gamma = \alpha, \beta); \quad R = \frac{GF}{2\alpha_1(kEJ)^{1/2}}, \quad S = \frac{2I}{m}\left(\frac{k}{EJ}\right)^{1/2}$$

Here G is the shear modulus, F is the section area, I is the moment of inertia per unit length of the beam relative to the neutral plane, and α_1 is a coefficient which depends on the shape of the section (for a rectangular section $\alpha_1 = 1, 2$).

Substituting (1.2) into (1.1) and putting $\eta = \tau - \tau_1$ we obtain

$$y = y_0\mu, \quad y_0 = \frac{P}{2(4k^3EJ)^{1/4}} \tag{1.3}$$

$$\mu = \frac{2}{\pi}\int_0^\infty \frac{1}{2SD_1^{1/2}} [f_\alpha J_\alpha(\tau, \lambda) - f_\beta J_\beta(\tau, \lambda)]\, d\lambda$$

$$J_\gamma(\tau, \lambda) = \int_0^\tau \cos\lambda\left[\zeta + v_{01}\eta \pm \frac{1}{2}w_1\eta(2\tau - \eta)\right]\sin\eta\gamma\, d\eta = \tag{1.4}$$

$$\nu\int_0^{\pm\Delta v_1} \cos\lambda\left[\zeta + v_1\nu\eta_1 \mp \frac{1}{2}\nu\eta_1^2\right]\sin\nu\eta_1\gamma\, d\eta_1 = \operatorname{Re}\left[\frac{\nu\exp(i\lambda\zeta)}{2i}(J_{2\gamma} - J_{1\gamma})\right]$$

$$J_{k\gamma} = \int_0^{\pm\Delta v_1} \exp[-(a\eta_1^2 + 2b_{k\gamma}\eta_1)]\,d\eta_1$$

$$\nu = 1/w_1, \quad v_1 = v_{01} \pm w_1\tau, \quad \Delta v_1 = v_1 - v_{01}$$

$$\eta_1 = \eta/\nu, \quad a = \pm{}^1/_2 i\nu\lambda$$

$$b_{k\gamma} = -{}^1/_2 i\nu\,[\lambda v_1 + (-1)^k\gamma] \quad (\gamma = \alpha, \beta;\ k = 1, 2)$$

The integrals $J_{k\gamma}$ are expressed in terms of the probability integral /6/

$$J_{k\gamma} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,[F(z_{k\gamma}) - \exp(z_{k\gamma}^2 - z_{0k\gamma}^2)\,F(z_{0k\gamma})] \tag{1.5}$$

$$F(z) = \exp z^2 \operatorname{erfc}(z) \quad z_{k\gamma} = -\exp\left(\mp\frac{\pi i}{4}\right)\sqrt{\frac{\nu}{2\lambda}}\,i\,[\lambda v_1 + (-1)^k\gamma] \tag{1.6}$$

The quantity $z_{0k\gamma}$ is obtained from $z_{k\gamma}$ by replacing v_1 by v_{01}.

We use well-known expansions of the probability integral /6/ in calculating $F(z)$.

Let us examine the case of uniformly accelerated motion of the force P over the beam in more detail. To perform the numerical computations, especially for large values of ν, it is important to determine the position of the points $z_{k\gamma}$, $z_{0k\gamma}$ $(\gamma = \alpha, \beta)$ in the complex plane for different values of the parameters v_1, v_{01} and the variable of integration λ. According to (1.6), the points $z_{2\gamma}$ and $z_{02\gamma}$ lie in the third quadrant of the complex plane for $\lambda \geqslant 0$. To improve the convergence of the asymptotic series we apply the symmetry relationship /6/ $F(-z) = 2\exp z^2 - F(z)$ for the quantities $F(z_{2\gamma})$ and $F(z_{02\gamma})$ in (1.5). Let us consider the quantity $z_{1\alpha}$. This point will lie in the right half-plane for all $\lambda \geqslant 0$ if $\alpha > \lambda v_1$. This last requirement is satisfied for fairly small values of v_1. The boundary value of the dimensionless velocity v_1^* is found by equating $z_{1\alpha}$ to zero and requiring that the root λ be multiple. The condition $z_{1\alpha} = 0$ reduces to the form

$$c\lambda^4 - d\lambda^2 + 4 = 0$$

$$c = (1 - Sv_1^2)\left(1 - \frac{v_1^2}{R}\right), \qquad d = v_1^2\left(4 + \frac{S}{R}\right) - \frac{1}{R}$$

We obtain from the condition of multiplicity of the root

$$v_1^* = \left\{\frac{S}{R^2} - \frac{4}{R} - 8S + \left[\left(\frac{S}{R^2} - \frac{4}{R} - 8S\right)^2 + \left(16 - \frac{1}{R^2}\right)\left(4 - \frac{S}{R}\right)^2\right]^{1/2}\right\}^{1/2}\left|4 - \frac{S}{R}\right|^{-1} \tag{1.7}$$

The value of v_1^* obtained as a result agrees with the value of the critical velocity v_1 resulting in unboundedness of the stationary solution corresponding to uniform motion of the force /5/. The positive root of the equation $z_{1\alpha}(\lambda) = 0$ will be $\lambda = \lambda^* = ({}^1/_2 d/c)^{1/2}$ where c and d are determined for $v_1 = v_1^*$. As has been shown in /5/, $v_1^* < R^{1/2}$, $v_1^* < S^{-1/2}$, $v_1^* \leqslant 1$. The quantity $R^{1/2}$ corresponds to the shear wave velocity in the beam and $S^{-1/2}$ to the tension-compression wave velocity. We note that the quantity v_1^* results in a positive root λ^* under the additional requirement /5/

$$R > 2\,[S + (S^2 + 16)^{1/2}]^{-1} \tag{1.8}$$

If v_1 exceeds v_1^* but is less than $R^{1/2}$, then for two values of λ determined from the equation

$$\alpha - \lambda v_1 = 0 \tag{1.9}$$

transition of the point $z_{1\alpha}$ from the right half-plane to the left occurs, and vice-versa. The appropriate values of λ will be

$$\lambda_{1,2} = \left(\frac{d \mp D^{1/2}}{2c}\right)^{1/2}, \quad D = d^2 - 16c \tag{1.10}$$

As $v_1 \to R^{1/2}$ we have $\lambda_2 \to \infty$, $\lambda_1 \to 2/(4R + S - R^{-1})^{1/2}$. If v_1 exceeds $R^{1/2}$ we have just one positive root of (1.9) that agrees in form with λ_1 according to (1.10).

The remark made above that refer to the behaviour of the quantity $z_{1\alpha}$ remain valid even for the quantity $z_{01\alpha}$ with v_1 replaced by v_{01}. We will consider accelerated motion under the condition that $v_{01} < v_1^*$. Then the point $z_{01\alpha}$ will be in the right half-plane for all $\lambda \geqslant 0$. To evaluate the component $F(z_{1\alpha})$ for $v_1^* < v_1 < R^{1/2}$ and $\lambda_1 < \lambda < \lambda_2$, as well as for $v_1 \geqslant R^{1/2}$ and $\lambda > \lambda_1$, we use the symmetry relationship already mentioned and subsequent application of standard expansions. In connection with the quantity $J_{1\beta}$ the behaviour of the points $z_{1\beta}$ and $z_{01\beta}$ should be considered. The point $z_{1\beta}$ $(z_{01\beta})$ is in the right half-plane for $\lambda \geqslant 0$ if $v_1 < S^{-1/2}$ $(v_{01} < S^{-1/2})$. If $v_1 > S^{-1/2}$ $(v_{01} > S^{-1/2})$, then a point λ of the passage of $z_{1\beta}$ $(z_{01\beta})$ from the right to the left half-plane occurs; the expression for this point agrees with λ_2 according to (1.10).

Assuming that (1.8) is satisfied, we write the final expression for the quantity $\mu(\zeta, v_1)$ $(v_{01} < v_1^*)$

$$\mu = \frac{1}{\pi}\int_0^\infty \Phi(\lambda)\,d\lambda \quad (v_1 \leqslant v_1^*) \tag{1.11}$$

$$\mu = \frac{1}{\pi}\left\{S_1 + S_2 + \int_{\lambda_2}^\infty \Phi(\lambda)\,d\lambda\right\} \quad (v_1^* < v_1 < R^{1/2}) \tag{1.12}$$

$$\mu = \frac{1}{\pi}\left\{S_1 + \int_{\lambda_1}^\infty [\Phi_\alpha(\lambda) + \Phi_1(\lambda)]\,d\lambda\right\} \quad (R^{1/2} \leqslant v_1 \leqslant S^{-1/2}) \tag{1.13}$$

$$\mu = \frac{1}{\pi}\left\{S_1 + S_2 + \int_{\lambda_2}^\infty [\Phi_2(\lambda) + \Phi_\alpha(\lambda) - \Phi_\beta(\lambda)]\,d\lambda\right\} \quad (v_1 > S^{-1/2}) \tag{1.14}$$

Here

$$S_1 = \int_0^{\lambda_1} \Phi(\lambda)\,d\lambda, \quad S_2 = \int_{\lambda_1}^{\lambda_2} [\Phi_\alpha(\lambda) + \Phi_1(\lambda)]\,d\lambda \tag{1.15}$$

$$\Phi(\lambda) = \frac{1}{2S}\left(\frac{\pi\nu}{2\lambda D_1}\right)^{1/2} \operatorname{Re}\left\{\exp\left[i\left(\lambda\zeta + \frac{\pi}{4}\right)\right](\psi_\alpha - \psi_\beta)\right\},$$

$$\psi_\gamma = f_\gamma[F(z_{1\gamma}) + F(-z_{2\gamma}) - \exp(z_{1\gamma}^2 - z_{01\gamma}^2)F(z_{01\gamma}) - \exp(z_{2\gamma}^2 - z_{02\gamma}^2)F(-z_{02\gamma})]$$

$$\Phi_\gamma(\lambda) = \frac{1}{S}\left(\frac{\pi\nu}{2\lambda D_1}\right)^{1/2} f_\gamma \operatorname{Re}\left\{\exp\left[z_{1\gamma}^2 + i\left(\lambda\zeta + \frac{\pi}{4}\right)\right]\right\} \quad (\gamma = \alpha, \beta)$$

$\Phi_1(\lambda)$ is obtained from $\Phi(\lambda)$ by replacing $F(z_{1\alpha})$ by $-F(-z_{1\alpha})$, $\Phi_2(\lambda)$ from $\Phi_1(\lambda)$ by replacing $F(z_{1\beta})$ by $-F(-z_{1\beta})$.

No difficulties should be encountered in performing calculations using (1.11)-(1.15) if the parameter ν does not take large values. However, it is the case of large values of ν (small accelerations), for which resonance growth of the deflections and stresses should occur at the critical velocities, that is of greatest theoretical and practical interest. The calculational difficulties for large values of ν are associated with rapid oscillations of the functions $\Phi_\gamma(\lambda)$. Moreover, the neighbourhoods of the points λ_1 and λ_2 at which the quantities $z_{1\gamma}$ vanish require attention. At the points λ_1 and λ_2 the functions $\Phi(\lambda)$, $\Phi_1(\lambda)$, $\Phi_2(\lambda)$ grow as the parameter ν increases since $F(0) = 1$ while growth of the functions already does not occur for $\lambda \neq \lambda_1, \lambda_2$. Because of the abrupt change in the integrands the mentioned neighbourhoods must be shrunk with a simultaneous increase in the number of nodes of the quadrature formula. The difficulties associated with the rapid oscillations are overcome by application of a Filon-type quadrature formula.

Note that components containing $z_{0k\gamma}$ can be discarded in the solutions (1.11)-(1.15) for large values of ν. These components represent the transient and damp out rapidly as the quantity Δv_1 increases, which is assured by the oscillating factor $\exp(z_{k\gamma}^2 - z_{0k\gamma}^2)$ (an additional exponentially decreasing factor appears when dissipation is taken into account).

2. Asymptotic behaviour of the solution as $\nu \to \infty$. The case $v_1 < v_1^*$. Only the first terms of the asymptotic expansion of the probability integral can remain in (1.11); consequently, we arrive at a stationary solution corresponding to the running value of the dimensionless velocity v_1 /5/.

The case $v_1 = v_1^*$. Now for a small neighbourhood of the point of application of the force an increase in the deflections is characteristic as the parameter ν increases. The main term of the asymptotic form is generated by a neighbourhood of the point λ^*, where $z_{1\alpha}$ vanishes together with its derivative

$$z_{1\alpha} \approx (\nu/\lambda^*)^{1/2} 2^{-3/2} \alpha''(\lambda^*) \exp(\pi i/4)(\lambda - \lambda^*)^2 \tag{2.1}$$

Making the substitution

$$\sigma = \nu^{1/4} 2^{-3/4} [\alpha''(\lambda^*)]^{1/2} (\lambda^*)^{-1/4} (\lambda - \lambda^*) \tag{2.2}$$

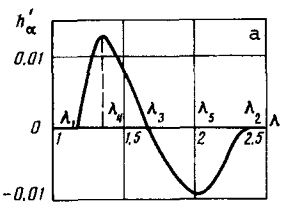
for the integration over the neighbourhood mentioned and keeping only the component with $F(z_{1\alpha})$ in the solution (1.11), we arrive at the asymptotic representation

$$\mu(\zeta, v_1^*) \approx \frac{1}{4S}\Gamma\left(\frac{1}{4}\right)\left(\frac{\nu}{2\lambda^*}\right)^{1/4} f_\alpha(\lambda^*)[\pi D_1(\lambda^*)\alpha''(\lambda^*)]^{-1/2} \times \cos\left(\lambda^*\zeta + \frac{\pi}{8}\right) \tag{2.3}$$

The case $v_1^* < v_1 < R_1^{1/2}$. To construct the asymptotic representation of the solution, the neighbourhood of the points λ_1 and λ_2 should be studied for the integration of $\Phi(\lambda)$ and $\Phi_1(\lambda)$, as should the stationary points of the integral of $\Phi_\alpha(\lambda)$. For the investigations of the stationary points we write

$$z_{1\alpha}^2 + i\lambda\zeta = i\nu[h_\alpha(\lambda) + \lambda\zeta_\nu], \quad h_\alpha(\lambda) = \frac{(\lambda v_1 - \alpha)^2}{2\lambda}, \quad \zeta_\nu = \frac{\zeta}{\nu} \tag{2.4}$$

The behaviour of the derivative $h_\alpha'(\lambda)$ is shown in Fig.1a ($v_1 = 1$) in the case under consideration (the graphs in Fig.1 correspond to the values $R^{1/2} = 1.5$, $S^{-1/2} = 4$, and $v_1^* = 0.932$). The points $\lambda_1, \lambda_2, \lambda_3$ are stationary and the second derivative of the function $h_\alpha(\lambda)$ vanishes at the points λ_4 and λ_5.



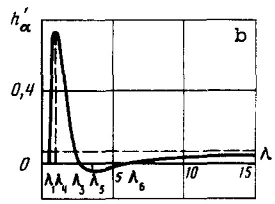


Fig.1

Without going into detail, we say that for small $|\zeta|$ the complete asymptotic representation is determined by the sum of the stationary solution μ_1 and the contribution of the stationary point λ_3

$$\mu(\zeta, v_1) \approx \mu_1 + \frac{f_\alpha(\lambda_3)\cos[\nu h_\alpha(\lambda_3) + \lambda_3\zeta]}{S[\lambda_3 D_1(\lambda_3)\,|h_\alpha''(\lambda_3)|]^{1/2}} \tag{2.5}$$

It is clear from Fig.1 that values ζ_ν can be mentioned for which the most intense increase in the deflections should be expected as ν grows. Indeed, for $\zeta_\nu = \zeta_{\nu 1} = -h_\alpha'(\lambda_4)$ and $\zeta_\nu = \zeta_{\nu 2} = -h_\alpha'(\lambda_5)$ at the points λ_4 and λ_5 we will have second-order stationary points that will result in asymptotic representations of the form

$$\mu(\zeta_{\nu 1}, v_1) \approx \frac{\nu^{1/6}}{S}\left(\frac{3}{2\pi\lambda_4 D_1(\lambda_4)}\right)^{1/2} f_\alpha(\lambda_4)\,\Gamma\left(\frac{4}{3}\right) \times \left(\frac{6}{|h_\alpha'''(\lambda_4)|}\right)^{1/3} \cos\left[\nu h_\alpha(\lambda_4) + \nu\lambda_4\zeta_{\nu 1} + \frac{\pi}{4}\right] \tag{2.6}$$

The formula for $\mu(\zeta_{\nu 2}, v_1)$ is obtained from (2.6) by replacing $\zeta_{\nu 1}$ by $\zeta_{\nu 2}$ and λ_4 by λ_5.

The case $v_1 = R^{1/2}$. As the parameter v_1 approaches the quantity $R^{1/2}$ from below, the point λ_2 goes to infinity. The estimation of the contribution of the infinitely remote point λ_2 represents the greatest complexity in constructing the asymptotic representation of the solution. For $\zeta = 0$ the replacement $\sigma = \lambda^{-1/2}$ is made, which removes the point mentioned to the origin. Components that increase as $\nu^{1/2}$ as ν increases occur during the integration of each of the functions Φ_α and Φ_1 in the neighbourhood of $\sigma = 0$; however, these components are mutually annihilated in satisfying condition (1.8). Consequently, for $\zeta = 0$ the asymptotic form (2.5) is valid, where the component μ_1 for $\zeta = 0$ vanishes for $\zeta = 0$. Calculations show that the asymptotic form (2.5) actually remains valid for $\zeta \leqslant 0$.

We note that in the case when condition (1.8) is not satisfied the quantity μ increases $\nu^{1/2}$ as ν increases for $\zeta = 0$, which corresponds to the result for a string or a rod in which only shear strains occur /2/. In this case the two critical velocities v_1^* and $R^{1/2}$ merge into one $R^{1/2}$.

The case $R^{1/2} < v_1 < S^{-1/2}$. The form of the dependence h_α' is shown in Fig.1b ($v_1 = 1.8$). For a small neighbourhood of the point of application of the force the asymptotic form contains the stationary solution μ_1 /5/ to which the contribution of the stationary points λ_3 and λ_6

$$\mu \approx \mu_1 + \frac{f_\alpha(\lambda_3)\cos[\nu h_\alpha(\lambda_3) + \zeta\lambda_3]}{S[\lambda_3 D_1(\lambda_3)\,|h_\alpha''(\lambda_3)|]^{1/2}} - \frac{f_\alpha(\lambda_6)\sin[\nu h_\alpha(\lambda_6) + \zeta\lambda_6]}{S[\lambda_6 D_1(\lambda_6)\,h_\alpha''(\lambda_6)]^{1/2}} \tag{2.7}$$

should be added for v_1 close to $R^{1/2}$ ($h_\alpha'(\lambda_5) < 0$).

If $h_\alpha'(\lambda_5) > 0$ (as v_1 approaches $S^{-1/2}$), then the complete asymptotic form is determined by μ_1.

We also considered a point on the beam with abscissa $\zeta_\nu = \zeta_{\nu 3} = -h_\alpha'(\infty) = -{}^1/_2(v_1 - R^{1/2})^2$. In this case there is a stationary point for the integral of the function $\Phi_\alpha(\lambda)$ located at infinity. It turns out that when condition (1.8) is satisfied the deflections at the point $\zeta_{\nu 3}$ decrease as ν increases; if this condition is not satisfied, then as in the case of a string /2/, an increase in the deflections occurs at the point $\zeta_{\nu 3}$ as ν increases for velocities close to $R^{1/2}$. Note that the derivative of the deflections with respect to the variable ζ becomes infinite for the point $\zeta_{v3}+0$.

The case $v_1 \geqslant S^{-1/2}$. Now it is necessary to take account of the appearance of a point λ_2 on the real axis where the quantity $z_{1\beta}$ vanishes. The point λ_2 will be stationary for an integral of the function $\Phi_\beta(\lambda)$ for $v_1 > S^{-1/2}$ and an abrupt change in the function $\Phi_2(\lambda)$ occurs in the neighbourhood of this point for large values of ν.

For a small neighbourhood of the point of application of the force, the asymptotic form is reduced to the stationary solution for uniform motion of the force /5/. In addition to the points ζ_{v1}, ζ_{v2}, ζ_{v3}, consider earlier the point ζ_{v4} determined by the value of $\zeta_{v4} = -h_\beta'(\lambda_7)$ should be considered (for $\lambda=\lambda_7$ the quantity h_β' reaches a maximum). According to the method of stationary phase, the neighbourhood of the point λ_7 yields a contribution to the quantity μ for $\zeta_v = \zeta_{v4}$ that increases as $\nu^{1/4}$, however because of the rapid decrease in the function $\Phi_\beta(\lambda)$ as λ increases, the quantity $\mu(\zeta_{v4}, v_1)$ remains small even for very large values of ν.

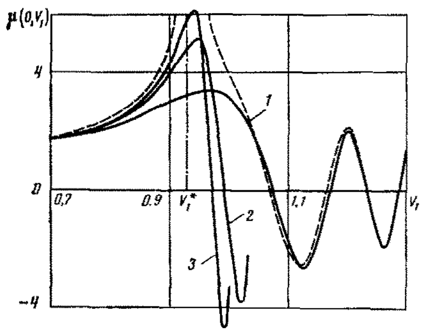


Fig.2

3. Results of numerical calculations.

The examples presented below for numerical calculations illustrate the dynamic effects associated with passages through the critical velocities. The calculations were performed for values of the parameters $R^{1/2}=1.5$; $S^{-1/2}=4$, and $v_1^*=0.932$. The graphs in Fig.2 demonstrate the passage through the first critical velocity v_1^* for $\zeta=0$. Curves *1, 2, 3* correspond to the values $\nu=100$, 500, and 1000. The dashed curves correspond to asymptotic representations for the cases $v_1<v_1^*$ and $v_1^*<v_1<R^{1/2}$ which approximate the solution outside a small neighbourhood of the critical velocity v_1^* quite well. For $v_1=v_1^*$ the asymptotic form (2.3) should be used, which results in several correct significant figures even for $\nu=100$, as calculations show.

Presented in Fig.3 for the case $\nu=1000$ are results corresponding to the points ζ_{v1} (curve *1*) and ζ_{v2} (curve *2*). Also given here is a comparison with the asymptotic (dashes). We note that as the parameter v_1 tends to the value v_1^* the quantities ζ_{v1} and ζ_{v2} tend to zero, consequently for $v_1=v_1^*$ the ordinates of the graphs in Fig.3 agree with the corresponding ordinate in Fig.2. For velocities v_1 larger than v_1^* the values of the deflections at the point ζ_{v1} turn out to be greater than for $\zeta=0$.

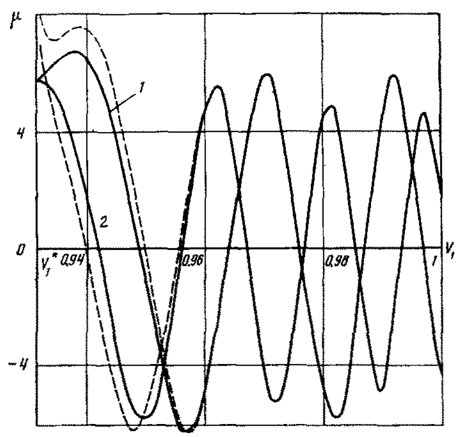


Fig.3

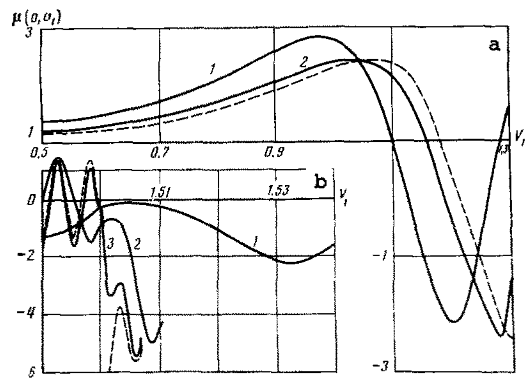


Fig.4

We note that the behaviour of the deflections of a Timoshenko beam during the passage through the first critical velocity is analogous to the results for the Bernoulli-Euler beam /4/. Fig.4a shows graphs of the quantity $\mu(0, v_1)$ for $\nu=50$. The dashed curve corresponds to the Bernoulli-Euler beam ($R\to\infty$, $S\to 0$), and curves *1, 2* correspond to the parameters $R=2.25$, $S=1/16$ and $R=10$, $S=1/64$. As is seen, the results change insignificantly for the range of variation of the parameters R, S considered.

Fig.4b shows the passage through the second critical velocity $R^{1/2}$, where deflections at the point of application of the force $\zeta=0$ are considered. The parameters R and S are as before ($R^{1/2}=1.5$, $S^{-1/2}=4$). Curves *1, 2, 3* correspond to the values $\nu=100$, 500, 1000. The deflections increase as the parameter ν increases, where the greatest values are achieved after the time $v_1 = R^{1/2}$. For the case $\nu = 1000$ the graph of the asymptotic representation (2.5) is shown by dashes for $v_1 < R^{1/2}$, and (2.7) for $v_1 > R^{1/2}$. As already mentioned, the asymptotic form (2.5) turns out to be valid even for $v_1 = R^{1/2}$.

For a further increase in the velocity v_1 and large values of the parameter ν the solution of the problem approaches the stationary solution μ_1 which, if only the beam deflections are kept in mind, does not exhibit resonance effects when the velocity passes through the value $S^{-1/2}$. Consequently, the numerical data reflecting the passage through the velocity $S^{-1/2}$ are not presented.

In conclusion, we note that the method elucidated above can be used even to study the stresses in a beam. Thus, in place of the function u in (1.2) it is sufficient to apply the expression for the bending moments under the action of an instantaneous impulse on a beam /5, 7/ when considering the bending moments. An increase in the bending moments as the parameter ν increases will occur on passing through all three critical velocities $v_1{}^*$, $R^{1/2}$, $S^{-1/2}$.

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SOLITARY LONGITUDINAL WAVES IN AN INHOMOGENEOUS NON-LINEARLY ELASTIC ROD*

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The solution of the Cauchy problem for the equation of longitudinal displacement wave propgation in an infinitely long elastic rod is considered taking the physical and geometric non-linearities of the material, the wave dispersion, and inhomogeneities of the second and third order elastic moduli into account. A slow change in the elastic moduli in the wave propagation direction results in a perturbation of the equation of the problem solvable by the method of multiscale decomposition. It is shown that for certain initial data the solution of the problem is a soliton in the longitudinal displacement velocity. The soliton parameters are determined by the elastic moduli of the material, and its propagation over the rod is accompanied by a low-amplitude long-wave (plateau). Relations are derived between the elastic moduli for which the soliton amplitude remains constant or the plateau is not formed behind the main impulse. Under other initial conditions the Cauchy problem is solved numerically, and shaping of the solitary waves is investigated. Soliton

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